Example 1. We know that $f: x \mapsto x^2$ is cts. Here we show that it is not uniformly cts. Take $\varepsilon = 1/2$. Let $\delta > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$. Define x, u:

$$x = N + \tfrac{1}{N}, \quad u = N.$$

Then $|x - u| = \frac{1}{N} < \delta$ but

$$|f(x) - f(u)| = \left(N + \tfrac{1}{N}\right)^2 - (N)^2 > 2N \cdot \tfrac{1}{N} = 2 > \varepsilon$$

so f <u>not</u> unif. cts.

Example 2. $f: x \mapsto \frac{1}{x^2}$ $\forall x > 0$ (or $x \neq 0$).

Then f is not uniformly cts (although it is cts)

Hint. Consider $x = \frac{1}{n}$ & $u = \frac{1}{2n}$.

Question How about $x \mapsto \frac{1}{\sqrt{x}}$ ($\forall x > 0$)?
— same ans.

Example 3. Let $f(x) = \frac{1}{x^2+1}$ $\forall x$. Then f is uniformly cts.

Hint: $$\left| \frac{1}{x^2+1} - \frac{1}{u^2+1} \right| = \frac{|u^2 - x^2|}{(x^2+1)(u^2+1)} \leq |u-x| \cdot \frac{|u| + |x|}{\cdots}$$

$$\leq |u - x| \left(\frac{|u|}{(x^2+1)(u^2+1)} + \frac{|x|}{(x^2+1)(u^2+1)} \right)$$

$$\le |u-x|\left(\frac{|u|}{u^2+1} + \frac{|x|}{x^2+1}\right) \le |u-x|(1+1)$$

Similarly, $x \mapsto \frac{1}{x}$ is uniformly cts on $[\varepsilon, \infty)$ $\forall \varepsilon > 0$.

Th (Lipschitz Function) Let $f: A \to \mathbb{R}$ be Lipschitz-continuous in the sense that $\exists$ a constant $k > 0$ s.t.

$$|f(x) - f(u)| \le k|x-u| \quad \forall x, u \in A$$

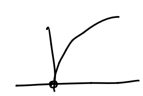
slope of the graph of f is bounded by k

Then f is uniformly cts.

Pf. Easy Ex

Remark. But not the converse, i.e.

unif cts $\not\Rightarrow$ Lipschitz

(e.g. $x \mapsto \sqrt{x}$ on $[0,1]$ )

Th (Uniform Continuity Th).

Let $A \subseteq \mathbb{R}$ be bounded & "closed" in the sense that

$$x = \lim_n a_n, \ a_n \in A \ \forall n \implies x \in A$$

(e.g. A is $[a,b]$ for some $a, b \in \mathbb{R}$). Let $f: A \to \mathbb{R}$ be cts. Then f is unif. cts. (the converse is of course always true).

Proof. Suppose f is not uniformly cts: $\exists\ \varepsilon>0$ s.t. $\forall\ \delta>0\ \exists\ x,u\in A$ with $|x-u|<\delta$ but $|f(x)-f(u)|\geq\varepsilon$; in particular, for this $\varepsilon>0$ one has that $\forall\ n\in\mathbb{N}\ \exists$ $x_n,u_n\in A$ with $|x_n-u_n|<\frac{1}{n}$ but $|f(x_n)-f(u_n)|\geq\varepsilon$. (#)

In this way, one has two seq (x_n) & (u_n) in A satisfying (#) $\forall\ n$. By assumption (that A is bounded & closed) and by B-W Theorem, $\exists$ a subseq (x_{n_k}) of (x_n) convergent with $x:=\lim_k x_{n_k}\in A$. Since

$$|x_{n_k}-u_{n_k}|<\frac{1}{n_k}\to 0\ (\text{as } k\to\infty)$$

it follows that $x=\lim_k u_{n_k}$. Since f is cts at x it follows from the seq. criterion that $f(x)=\lim_k f(u_{n_k})$ & $f(x)=\lim_k f(x_{n_k})$

and so $\lim_k\big(f(x_{n_k})-f(u_{n_k})\big)=f(x)-f(x)=0$

contradicting (#) (as $|f(x_{n_k})-f(u_{n_k})|\geq\varepsilon\ \forall\ k$).